Proposition de cours M2 MPA(2024-2025)

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Laurent Stolovitch : **Introduction to Nash-Moser theory and applications in dynamical systems and geometry.**

Both in geometry and in dynamical systems, there are two main aspects of the (local) analysis of the objects : The first one is the behaviour at points where "nothing happens" : those points where a geometrical structure or a dynamical system are in an appropriate sense uniform.The other one is the behaviour at points where "things go wrong" : the geometry or the behaviour of the dynamical system changes in a drastic way. Such a point bears different names–depending on the field–called *singularity* or *catastrophe*. The aim of these lectures is to provide key concepts/tools for deciphering what is going on at such a bad point. We aim at defining a local model that is supposed to reflect the very nature of the considered object at the bad point. This *normal form* is obtained in a precise way by the action of a local group of transformations (fixing the bad point) on the original object. Transformation to Jordan normal form of matrices is a "baby" problem of that kind.
A wide range of these problems can be formulated as a solution of some functional equation $\mathcal{F} = 0$ defined on some functional spaces which are not in general Banach spaces but rather Fréchet spaces. In these spaces, the usual implicit function theorem does not hold as the derivative of $\mathcal{F}$ may be invertible (with bounded inverse) at a point and not in neighborhood of it. In this lecture, we shall explain the so called "Hard implicit function theorem", known also as Nash-Moser theorem or Newton method that helps handling these problems. These techniques are very usuful for a wide range of problems, from geometry to PDE's and dynamical systems. We shall illustrate them in the study of linearization or normal form problems of (germs of) holomorphic dynamical systems (diffeomorphisms or vector fields) at a singular point (i.e a fixed point). This requires to develop the concepts of *resonances* and *small divisors*. We shall show the existence of an infinite number of resonances is a serious obstruction to obtaining a good holomorphic model. Finally, we shall show how dynamics alike phenomena can infer deep understanding of the geometry of neighborhoods of embedded compact complex manifolds in some complex manifolds.
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